

# A NOTE ON CONVERGENCE OF LOW ENERGY CRITICAL POINTS OF NONLINEAR ELASTICITY FUNCTIONALS, FOR THIN SHELLS OF ARBITRARY GEOMETRY

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**ABSTRACT.** We prove that the critical points of the 3d nonlinear elasticity functional on shells of small thickness  $h$  and around the mid-surface  $S$  of arbitrary geometry, converge as  $h \rightarrow 0$  to the critical points of the von Kármán functional on  $S$ , recently derived in [10]. This result extends the statement in [16], derived for the case of plates when  $S \subset \mathbb{R}^2$ . We further prove the same convergence result for the weak solutions to the static equilibrium equations (formally the Euler-Lagrange equations associated to the elasticity functional). The convergences hold provided the elastic energy of the 3d deformations scale like  $h^4$  and the external body forces scale like  $h^3$ .

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## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Since the beginning of research in nonlinear elasticity, a major topic has been the derivation of lower dimensional theories, appropriately approximating the three dimensional theory on structures which are thin in one or more directions (such as beams, rods, plates or shells). Recently, the application of variational methods, notably the  $\Gamma$ -convergence [5], lead to many significant and rigorous results in this setting [9, 7]. Roughly speaking, a  $\Gamma$ -limit approach guarantees the convergence of minimizers of a sequence of functionals, to the minimizers of the limit. However, it does not usually imply convergence of the possibly non-minimizing critical points (the equilibria) and hence other tools must be applied to study this problem.

In this note, following works [13, 14, 16] in which beams, rods and plates were analyzed, we study critical points of the 3d nonlinear elasticity functional on a thin shell of arbitrary geometry, in the von Kármán scaling regime. A  $\Gamma$ -convergence result in this framework was recently derived in [10], providing the natural from the minimization point of view generalization of the von Kármán functional [7] to shells. In analogy with the analysis done in [16] for plates, we now proceed to establish convergence of weak solutions to the (formal) Euler-Lagrange equations (1.11), as well as convergence of critical points of the 3d energy functionals (1.1), to the critical points of the

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functional obtained in [10]. As pointed out in [1] Problem 5, in general it is still unknown whether these two definitions of equilibria are equivalent.

We now introduce the basic framework for our results. We consider a 2-dimensional surface  $S$  embedded in  $\mathbb{R}^3$ , which is compact, connected, oriented, of class  $\mathcal{C}^{1,1}$ , and with boundary  $\partial S$  being the union of finitely many (possibly none) Lipschitz curves. A family  $\{S^h\}_{h>0}$  of shells of small thickness  $h$  around  $S$  is given through:

$$S^h = \{z = x + t\vec{n}(x); x \in S, -h/2 < t < h/2\}, \quad 0 < h < h_0.$$

By  $\vec{n}(x)$  we denote the unit normal to  $S$ , by  $T_x S$  the tangent space, and  $\Pi(x) = \nabla \vec{n}(x)$  is the shape operator on  $S$  (the negative second fundamental form). The projection onto  $S$  along  $\vec{n}$  is denoted by  $\pi$ . We assume that  $h < h_0$ , with  $h_0$  sufficiently small to have  $\pi$  defined on each  $S^h$ .

To a deformation  $u \in W^{1,2}(S^h, \mathbb{R}^3)$  we associate its elastic energy (scaled per unit thickness):

$$(1.1) \quad E^h(u) = \frac{1}{h} \int_{S^h} W(\nabla u).$$

The stored energy density  $W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty]$  is assumed to be  $\mathcal{C}^2$  in a neighborhood of  $SO(3)$ , and to satisfy the following normalization, frame indifference and nondegeneracy conditions:

$$(1.2) \quad \begin{aligned} \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3) \quad W(R) = 0, \quad W(RF) = W(F), \\ W(F) \geq C \text{dist}^2(F, SO(3)) \end{aligned}$$

(with a uniform constant  $C > 0$ ). Our objective is to describe the limiting behavior, as  $h \rightarrow 0$ , of critical points  $u^h$  to the following total energy functionals:

$$(1.3) \quad J^h(u) = E^h(u) - \frac{1}{h} \int_{S^h} f^h u,$$

subject to external forces  $f^h$ , where we assume that:

$$f^h(x + t\vec{n}) = h\sqrt{e^h} f(x) \det(\text{Id} + t\Pi)^{-1}, \quad f \in L^2(S, \mathbb{R}^3) \text{ and } \int_S f = 0.$$

Above,  $e^h$  is a given sequence of positive numbers obeying a prescribed scaling law. It can be shown [7, 10] that if  $f^h$  scale like  $h^\alpha$ , then the minimizers  $u^h$  of (1.3) satisfy  $E^h(u^h) \sim h^\beta$  with  $\beta = \alpha$  if  $0 \leq \alpha \leq 2$  and  $\beta = 2\alpha - 2$  if  $\alpha > 2$ . Throughout this paper we shall assume that  $\beta \geq 4$ , or more generally:

$$(1.4) \quad \lim_{h \rightarrow 0} e^h / h^4 = \kappa < +\infty,$$

which for  $S \subset \mathbb{R}^2$  corresponds to the von Kármán and the purely linear theories of plates, derived rigorously in [7].

In our recent paper [10], the  $\Gamma$ -limit of  $1/e^h J^h$  has been identified in the scaling range corresponding to (1.4), and for arbitrary surfaces  $S$ . It turns out that the elastic energy scaling  $E^h(u^h) \leq C e^h$  implies that on  $S$  the deformations  $u^h|_S$  must be close to some rigid motion  $\bar{Q}x + c$ , and that the first order term in the expansion of  $\bar{Q}^T(u^h|_S - c) - \text{id}$  with respect to  $h$ , is an element  $V$  of the class  $\mathcal{V}$  of *infinitesimal isometries* on  $S$  [17]. The space  $\mathcal{V}$  consists of vector fields  $V \in W^{2,2}(S, \mathbb{R}^3)$  for whom there exists a matrix field  $A \in W^{1,2}(S, \mathbb{R}^{3 \times 3})$  so that:

$$(1.5) \quad \partial_\tau V(x) = A(x)\tau \quad \text{and} \quad A(x)^T = -A(x) \quad \forall \text{a.e. } x \in S \quad \forall \tau \in T_x S.$$

Equivalently, the change of metric on  $S$  induced by the deformation  $\text{id} + hV$  is at most of order  $h^2$ , for each  $V \in \mathcal{V}$ .

When in (1.4)  $\kappa = 0$ , the limiting total energy is given by:

$$(1.6) \quad J(V, \bar{Q}) = \frac{1}{24} \int_S \mathcal{Q}_2(x, (\nabla(A\bar{n}) - A\Pi)_{tan}) \, dx - \int_S f \cdot \bar{Q}V \, dx, \quad \forall V \in \mathcal{V}, \bar{Q} \in SO(3).$$

The first term above measures the first order change in the second fundamental form  $\Pi$  of  $S$ , produced by  $V$ . The quadratic forms  $\mathcal{Q}_2(x, \cdot)$  are given as follows:

$$\mathcal{Q}_2(x, F_{tan}) = \min\{\mathcal{Q}_3(\tilde{F}); (\tilde{F} - F)_{tan} = 0\}, \quad \mathcal{Q}_3(F) = D^2W(\text{Id})(F, F).$$

The form  $\mathcal{Q}_3$  is defined for all  $F \in \mathbb{R}^{3 \times 3}$ , while  $\mathcal{Q}_2(x, \cdot)$  for a given  $x \in S$ , is defined on tangential minors  $F_{tan}$  of such matrices. Both forms depend only on the symmetric parts of their arguments and are positive definite on the space of symmetric matrices [6]. In the weak formulation of the Euler-Lagrange equations of (1.6) one naturally encounters the linear operators  $\mathcal{L}_3$  and  $\mathcal{L}_2(x, \cdot)$ , defined on matrix spaces  $\mathbb{R}^{3 \times 3}$  and  $\mathbb{R}^{2 \times 2}$  respectively, given by:

$$\forall F \in \mathbb{R}^{3 \times 3} \quad \mathcal{Q}_3(F) = \mathcal{L}_3 F : F \quad \text{and} \quad \mathcal{Q}_2(x, F_{tan}) = \mathcal{L}_2(x, F_{tan}) : F_{tan}.$$

For  $\kappa > 0$ , the  $\Gamma$ -limit (which is the generalization of the von Kármán functional [7] to shells), contains also a stretching term, measuring the total second order change in the metric of  $S$ :

$$(1.7) \quad J^{vK}(V, B_{tan}, \bar{Q}) = \frac{1}{2} \int_S \mathcal{Q}_2\left(x, B_{tan} - \frac{\kappa}{2}(A^2)_{tan}\right) + \frac{1}{24} \int_S \mathcal{Q}_2(x, (\nabla(A\bar{n}) - A\Pi)_{tan}) - \int_S f \cdot \bar{Q}V.$$

It involves a symmetric matrix field  $B_{tan}$  belonging to the *finite strain space*:

$$\mathcal{B} = \text{cl}_{L^2(S)} \left\{ \text{sym} \nabla w^h; w^h \in W^{1,2}(S, \mathbb{R}^3) \right\}.$$

The two terms in (1.7) correspond, in appearing order, to the stretching and bending energies of a sequence of deformations  $v^h = \text{id} + hV + h^2 w^h$  of  $S$ , which is induced by a first order displacement  $V \in \mathcal{V}$  and second order displacements  $w^h$  satisfying  $\lim_{h \rightarrow 0} \text{sym} \nabla w^h = B_{tan}$ . The crucial property of (1.7) is the one-to-one correspondence between the minimizing sequences  $u^h$  of the total energies  $J^h(u^h)$ , and their approximations (modulo rigid motions  $\bar{Q}x + c$ ) given by  $v^h$  as above with  $(V, B_{tan}, \bar{Q})$  minimizing  $J^{vK}$ , or  $(V, \bar{Q})$  minimizing  $J$  when  $\kappa = 0$ .

The purpose of this paper is to show that under the following extra assumption of [16]:

$$(1.8) \quad \forall F \in \mathbb{R}^{3 \times 3} \quad |DW(F)| \leq C(|F| + 1).$$

also the equilibria (possibly non-minimizing) of (1.1) converge to the equilibria of (1.7) or (1.6). The definition of an equilibrium of the 3d energy  $J^h$  may be understood in two different manners, corresponding to passing with the scaling  $\epsilon$  of a variation  $\phi$  to 0 outside or inside the integral sign. Namely, for a fixed  $h > 0$ , we may require that:

$$(1.9) \quad \forall \phi^h \in W^{1,2}(S^h, \mathbb{R}^3) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( J^h(u^h + \epsilon \phi^h) - J^h(u^h) \right) = 0.$$

or that:

$$(1.10) \quad \forall \phi^h \in W^{1,2}(S^h, \mathbb{R}^3) \quad \int_{S^h} DW(\nabla u^h) : \nabla \phi^h = \int_{S^h} f^h \phi^h.$$

The last condition is obtained by formal passing to the limit  $\epsilon \rightarrow 0$  under the integral sign in (1.9). Integrating by parts we also see that (1.10) is the weak formulation of the following fundamental balance law [1]:

$$(1.11) \quad \text{div} [DW(\nabla u^h)] + f^h = 0 \text{ in } S^h, \quad DW(\nabla u^h) \bar{n} = 0 \text{ on } \partial S^h,$$

where the operator  $\text{div}$  above is understood as acting on rows of the matrix field  $DW(\nabla u^h)$ . Whether the two definitions of equilibria (1.9) and (1.10) are equivalent, even for local minimizers

(without assuming extra regularity, e.g. their Lipschitz continuity) is an open problem of nonlinear elasticity, listed by Ball as Problem 5 in [1].

It turns out that the main convergence result described below follows with either (1.9) or (1.10). The reason is that the difference between these two definitions (after an appropriate scaling), converges to 0 with  $h$ , along particular sequences of variations  $\phi^h$ , which are however exactly the 3d variations recovered from the the variations of the 2d functional  $J^{vK}$  or  $J$ .

**Theorem 1.1.** *Assume (1.2) and (1.8). Let  $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$  be a sequence of deformations, satisfying:*

- (a) *the equilibrium equations (1.10) hold,*
- (b)  *$E^h(u^h) \leq C e^h$ , where  $e^h$  is the scaling with (1.4).*

*Then there exist a sequence  $Q^h \in SO(3)$ , converging (up to a subsequence) to some  $\bar{Q} \in SO(3)$ , and  $c^h \in \mathbb{R}^3$  such that for the normalized rescaled deformations:*

$$y^h(x + t\vec{n}) = (Q^h)^T u^h(x + h/h_0 t\vec{n}) - c^h$$

*defined on the common domain  $S^{h_0}$ , we have:*

- (i)  *$y^h$  converge in  $W^{1,2}(S^{h_0})$  to  $\pi$ .*
- (ii) *The scaled average displacements:*

$$(1.12) \quad V^h(x) = \frac{h}{\sqrt{e^h}} \int_{-h_0/2}^{h_0/2} y^h(x + t\vec{n}) - x \, dt$$

*converge (up to a subsequence) in  $W^{1,2}(S)$  to some  $V \in \mathcal{V}$ .*

- (iii)  *$h/\sqrt{e^h} \operatorname{sym} \nabla V^h$  converge (up to a subsequence) in  $L^2(S)$  to some  $B_{tan} \in \mathcal{B}$ .*
- (iv) *The triple  $(V, B_{tan}, \bar{Q})$  satisfies the Euler-Lagrange equations of the functional  $J^{vK}$ . That is, for all  $\tilde{V} \in \mathcal{V}$  with  $\tilde{A} = \nabla \tilde{V}$  given as in the formula (1.5), and all  $\tilde{B}_{tan} \in \mathcal{B}$ , there holds:*

$$(1.13) \quad \int_S \mathcal{L}_2 \left( x, B_{tan} - \frac{\kappa}{2} (A^2)_{tan} \right) : \tilde{B}_{tan} = 0,$$

$$(1.14) \quad -\kappa \int_S \mathcal{L}_2 \left( x, B_{tan} - \frac{\kappa}{2} (A^2)_{tan} \right) : (A\tilde{A})_{tan} \\ + \frac{1}{12} \int_S \mathcal{L}_2 (x, (\nabla(A\vec{n}) - A\Pi)_{tan}) : (\nabla(\tilde{A}\vec{n}) - \tilde{A}\Pi)_{tan} = \int_S f \cdot \bar{Q}\tilde{V},$$

*When  $\kappa = 0$  then the couple  $(V, \bar{Q})$  satisfies (1.14) for all  $\tilde{V} \in \mathcal{V}$ , which is the Euler-Lagrange equations of the functional (1.6).*

**Theorem 1.2.** *Theorem 1.1 remains true if in the assumption (a) the formal equilibrium equation (1.10) are replaced by the critical point condition (1.9).*

We prove Theorem 1.1 in section 2 and Theorem 1.2 in section 3. In section 4 we derive the third Euler-Lagrange equation (after the first two (1.13) and (1.14)), corresponding to variation in  $\bar{Q} \in SO(3)$ . We first notice that the limiting  $\bar{Q}$  necessarily satisfies the constraint of the average torque  $\tau(\bar{Q}) = \int_S f \times \bar{Q}x \, dx$  being 0. The main difficulty arises now from the fact that the variations must be taken inside  $SO(3)$  in a way that this constraint remains satisfied. Assuming that such variations exist, we establish the limit equation under the nondegeneracy condition that  $Q^h$  approach  $\bar{Q}$  along a direction  $U \in T_{\bar{Q}}SO(3)$  for which  $\partial_U \tau(\bar{Q}) \neq 0$ .

**Remark 1.3.** Condition (1.8) of [16] is of technical importance. Notice that, in view of (1.2) resulting in  $DW(F) = 0$  for all  $F \in SO(3)$ , (1.8) is equivalent to:

$$\forall F \in \mathbb{R}^{3 \times 3} \quad |DW(F)| \leq C \text{dist}(F, SO(3)).$$

Using the last assumption in (1.2), the above implies that:  $|DW(F)| \leq CW(F)^{1/2}$  for all  $F \in \mathbb{R}^{3 \times 3}$ . Hence, roughly speaking,  $W$  has a quadratic growth and we see that (1.8) is actually quite restrictive. Independent from our research, Mora and Scardia [15] has presently established a result complementary to ours where the requirement (1.8) is relaxed, while the equilibrium condition of (1.3) is understood in a different manner, related to Ball's inner variations and the Cauchy stress balance law [1].

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## 2. CONVERGENCE OF WEAK SOLUTIONS TO THE EULER-LAGRANGE EQUATIONS (EQUILIBRIA) OF THE 3D ENERGIES

We first gather the relevant information from [10]:

**Lemma 2.1.** [10] *Let  $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$  be a sequence of deformations of shells  $S^h$ . Assume (1.4) and let the scaled energies  $E^h(u^h)/e^h$  be uniformly bounded. Then there exists a sequence of matrix fields  $R^h \in W^{1,2}(S, \mathbb{R}^3)$  with  $R^h(x) \in SO(3)$  for a.e.  $x \in S$ , and there exists a sequence of matrices  $Q^h \in SO(3)$  such that:*

- (i)  $\|(Q^h)^T R^h - \text{Id}\|_{W^{1,2}(S)} \leq C\sqrt{e^h}/h$ .
- (ii)  $h/\sqrt{e^h}((Q^h)^T R^h - \text{Id})$  converges (up to a subsequence) to a skew-symmetric matrix field  $A$ , weakly in  $W^{1,2}(S)$ .

Moreover, there exists a sequence  $c^h \in \mathbb{R}^3$  such that for the normalized rescaled deformations:

$$y^h(x + t\vec{n}) = (Q^h)^T u^h(x + h/h_0 t\vec{n}) - c^h$$

defined on the common domain  $S^{h_0}$ , the following holds.

- (iii)  $y^h$  converge in  $W^{1,2}(S^{h_0})$  to  $\pi$ .
- (iv) The scaled average displacements  $V^h$ , defined in (1.12) converge (up to a subsequence) in  $W^{1,2}(S)$  to some  $V \in \mathcal{V}$ , whose gradient is given by  $A$ , as in (1.5).
- (v)  $h/\sqrt{e^h} \text{sym } \nabla V^h$  converge (up to a subsequence) in  $L^2(S)$  to some  $B_{\tan} \in \mathcal{B}$ .

The statements in Theorem 1.1 (i), (ii), (iii) are contained in the Lemma above. It therefore suffices to use the extra assumptions (1.10) and (1.8) to recover equations (1.13) and (1.14) as  $h \rightarrow 0$ .

We start by rewriting the equilibrium equation (1.10) in a more convenient form. Clearly, every variation  $\phi^h \in W^{1,2}(S^h, \mathbb{R}^3)$  can be by a change of variables expressed as:

$$(2.1) \quad \phi^h(x + t\vec{n}) = \psi(x + th_0/h\vec{n}),$$

for the corresponding  $\psi \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$ . Then, (1.10) becomes:

$$(2.2) \quad \begin{aligned} & h^2 \sqrt{e^h} \int_S f(x) \int_{-h_0/2}^{h_0/2} \psi(x + t\vec{n}) \, dt \, dx \\ &= h \int_S \int_{-h_0/2}^{h_0/2} \det(\text{Id} + th/h_0 \Pi) DW(\nabla u^h(x + th/h_0 \vec{n})) : \nabla \phi^h(x + th/h_0 \vec{n}) \, dt \, dx. \end{aligned}$$

Notice also that:

$$(2.3) \quad \nabla \phi^h(x + th/h_0 \vec{n}) = \nabla \psi(x + t\vec{n}) \cdot P(x + t\vec{n}),$$

where the matrix field  $P \in L^\infty(S^{h_0}, \mathbb{R}^3)$  has the following non-zero entries:

$$P(x + t\vec{n})_{tan} = (\text{Id} + th/h_0 \Pi(x))^{-1} (\text{Id} + t\Pi(x)), \quad \vec{n}^T P(x + t\vec{n}) \vec{n} = h_0/h.$$

In view of Lemma 2.1, define the matrix fields  $E^h, G^h \in L^2(S^{h_0}, \mathbb{R}^{3 \times 3})$ :

$$E^h = \frac{1}{\sqrt{e^h}} DW(\text{Id} + \sqrt{e^h} G^h), \quad G^h(x + t\vec{n}) = \frac{1}{\sqrt{e^h}} \left( (R^h)^T \nabla u^h(x + th/h_0 \vec{n}) - \text{Id} \right).$$

With this notation, recalling the frame invariance of  $W$  in (1.2) we get, for every  $F \in \mathbb{R}^{3 \times 3}$ :

$$\begin{aligned} \frac{1}{\sqrt{e^h}} DW(\nabla u^h(x + th/h_0 \vec{n})) : F &= \frac{1}{\sqrt{e^h}} DW(R^h(\text{Id} + \sqrt{e^h} G^h)) : F \\ &= \frac{1}{\sqrt{e^h}} DW(\text{Id} + \sqrt{e^h} G^h) : (R^h)^T F = R^h E^h : F. \end{aligned}$$

In particular, (2.2) becomes, after exchanging  $\psi$  to  $(Q^h)^T \psi$ , using (2.3) and dividing both sides by  $\sqrt{e^h}$ :

$$\begin{aligned} (2.4) \quad & h^2 \int_S f(x) \int_{-h_0/2}^{h_0/2} Q^h \psi \, dt dx \\ &= h \int_S \int_{-h_0/2}^{h_0/2} \det(\text{Id} + th/h_0 \Pi) \left[ (Q^h)^T R^h(x) E^h(x + t\vec{n}) \right] : \nabla \phi^h(x + th/h_0 \vec{n}) \, dt dx \\ &= h \int_S \int_{-h_0/2}^{h_0/2} \det(\text{Id} + th/h_0 \Pi) \left[ (Q^h)^T R^h E^h \right]_{TS} : [(\nabla_{tan} \psi)(\text{Id} + th/h_0 \Pi)^{-1} (\text{Id} + t\Pi)] \, dt dx \\ &\quad + h_0 \int_S \int_{-h_0/2}^{h_0/2} \det(\text{Id} + th/h_0 \Pi) ((Q^h)^T R^h E^h \vec{n}) \cdot \partial_{\vec{n}} \psi(x + t\vec{n}) \, dt dx, \end{aligned}$$

where  $\nabla_{tan}$  denotes gradient in the tangent directions of  $T_x S$ . The subscript  $TS$  stands for taking the  $3 \times 2$  minor of the matrix under consideration, for example:  $\nabla_{tan} \psi = [\nabla \psi]_{TS}$ .

**Lemma 2.2.** *The sequence  $G^h$  converges (up to a subsequence), weakly in  $L^2(S^{h_0}, \mathbb{R}^{3 \times 3})$  to an  $L^2(S^{h_0})$  matrix field  $G$ , whose tangential minor has the form:*

$$(2.5) \quad G(x + t\vec{n})_{tan} = \left( B_{tan} - \frac{\kappa}{2} (A^2)_{tan} \right) + \frac{t}{h_0} (\nabla(A\vec{n}) - A\Pi)_{tan}.$$

Moreover, if (1.8) holds, then:

- (i)  $E^h$  converges (up to a subsequence) weakly in  $L^2(S^{h_0}, \mathbb{R}^{3 \times 3})$  to the matrix field  $E = \mathcal{L}_3 G$ .
- (ii) The sequence  $(Q^h)^T R^h(x) E^h(x + t\vec{n})$  converges (up to a subsequence) to  $E$ , weakly in  $L^2(S^{h_0}, \mathbb{R}^{3 \times 3})$ .

*Proof.* The convergence of  $G^h$  and the formula (2.5) follow from Lemma 3.6 and Lemma 4.1 in [10]. Convergence in (i) is a consequence of Proposition 2.3 in [16], where the crucial role was played by the following equivalent form of the assumption (1.8):

$$\forall F \in \mathbb{R}^{3 \times 3} \quad |DW(\text{Id} + F)| \leq C|F|.$$

Finally, (iii) is an immediate consequence of (ii) in view of Lemma 2.1 (i) and the boundedness of  $(Q^h)^T R^h$  in  $L^\infty(S^{h_0})$ . ■

**Lemma 2.3.** *The matrix field  $E \in L^2(S^{h_0}, \mathbb{R}^{3 \times 3})$ , defined in Lemma 2.2 (i) satisfies the following properties, a.e. in  $S^{h_0}$ :*

- (i)  $E\vec{n} = 0$ .
- (ii)  $E^T = E$ , that is:  $E$  is symmetric.
- (iii)  $E_{tan}(x + t\vec{n}) = \mathcal{L}_2(x, G_{tan}(x + t\vec{n}))$ .

*Proof.* To prove (i), one needs to pass  $h \rightarrow 0$  in (2.4) and use Lemma 2.2 (ii) to obtain:

$$(2.6) \quad \int_S \int_{-h_0/2}^{h_0/2} \left( E(x + t\vec{n})\vec{n} \right) \partial_{\vec{n}}\psi(x + t\vec{n}) \, dt dx = 0.$$

Now, any vector field  $\phi \in L^2(S^{h_0}, \mathbb{R}^3)$  has the form  $\phi = \partial_{\vec{n}}\psi$ , where  $\psi(x + t\vec{n}) = \int_{-h_0/2}^t \phi(x + s\vec{n}) \, ds$ . Therefore (i) follows from (2.6).

By frame indifference (1.2) and the fact that  $W$  is minimized at Id, it follows that  $DW(F) = 0$  for all  $F \in SO(3)$ . It implies that for all  $H \in so(3)$  there holds  $\mathcal{L}_3 H = 0$ , and so:  $E : H = \mathcal{L}_3 G : H = \mathcal{L}_3 H : G = 0$ , proving (ii).

The assertion (iii) follows from  $E = \mathcal{L}_3 G$  and the reasoning exactly as in the proof of Proposition 3.2 [16].  $\blacksquare$

A more precise information, with respect to that in Lemma 2.3 (ii) is given by:

**Lemma 2.4.** *There holds:*

- (i)  $\| \text{skew } E^h \|_{L^1(S^{h_0})} \leq C\sqrt{e^h}$ .
- (ii)  $\lim_{h \rightarrow 0} \frac{1}{h} \| \text{skew } E^h \|_{L^p(S^{h_0})} = 0$ , for some exponent  $p \in (1, 2)$ .

*Proof.* By frame indifference (1.2) one has:  $0 = DW(F) : HF = DW(F)F^T : H$ , for all  $F \in \mathbb{R}^{3 \times 3}$  and all  $H \in so(3)$  (since  $HF$  is a tangent vector to  $SO(3)F$  at  $F$ ). We further obtain that  $DW(F)F^T$  is a symmetric matrix. Apply this statement pointwise to the matrix field  $F = \text{Id} + \sqrt{e^h}G^h$ :

$$\begin{aligned} 0 &= \frac{1}{\sqrt{e^h}} \left( DW(\text{Id} + \sqrt{e^h}G^h) (\text{Id} + \sqrt{e^h}(G^h)^T) - (\text{Id} + \sqrt{e^h}G^h) DW^T(\text{Id} + \sqrt{e^h}G^h) \right) \\ &= E^h - (E^h)^T + \sqrt{e^h} \left( E^h(G^h)^T - G^h(E^h)^T \right). \end{aligned}$$

Hence the claim in (i) is proved, as by Lemma 2.2:

$$\| \text{sym } (E^h(G^h)^T) \|_{L^1(S^{h_0})} \leq C \| E^h \|_{L^2(S^{h_0})} \| G^h \|_{L^2(S^{h_0})} \leq C.$$

Now, (ii) follows from (i) in view of the boundedness of  $E^h$  in  $L^2(S^{h_0})$ , (1.4), and through an interpolation inequality:

$$\frac{1}{h} \| \text{skew } E^h \|_{L^p(S^{h_0})} \leq \frac{1}{h} \| \text{skew } E^h \|_{L^1}^\theta \| \text{skew } E^h \|_{L^2}^{1-\theta} \leq C/h \sqrt{e^h}^\theta = C \left( \sqrt{e^h}/h^2 \right)^\theta h^{2\theta-1},$$

where  $1/p = \theta + (1 - \theta)/2$  and  $\theta \in (0, 1)$ . Clearly, the above converges to 0, when  $\theta > 1/2$ .  $\blacksquare$

Introduce now the two matrix fields  $\bar{E}, \hat{E} \in L^2(S, \mathbb{R}^3)$  given by the 0th and 1st moments of  $E$ :

$$\bar{E}(x) = \int_{-h_0/2}^{h_0/2} E(x + t\vec{n}) \, dt, \quad \hat{E}(x) = \int_{-h_0/2}^{h_0/2} t E(x + t\vec{n}) \, dt.$$

It easily follows by Lemma 2.3 (iii) and Lemma 2.2 that:

$$(2.7) \quad \bar{E}_{tan}(x) = \int_{-h_0/2}^{h_0/2} \mathcal{L}_2(x, G_{tan}(x + t\vec{n})) \, dt = \mathcal{L}_2 \left( x, B_{tan} - \frac{\kappa}{2} (A^2)_{tan} \right),$$

$$(2.8) \quad \hat{E}_{tan}(x) = \int_{-h_0/2}^{h_0/2} \mathcal{L}_2(x, tG_{tan}(x + t\vec{n})) \, dt = \frac{h_0}{12} \mathcal{L}_2(x, (\nabla(A\vec{n}) - A\Pi)_{tan}).$$

We will now use the fundamental balance (2.4) and the above formulas to recover the Euler-Lagrange equations (1.13), (1.14) in the limit as  $h \rightarrow 0$ .

**Proof of the first Euler-Lagrange equation (1.13).**

Use the variation of the form:  $\psi(x + t\vec{n}) = \phi(x)$  in (2.4), divide both sides by  $h$  and pass to the limit to obtain:

$$(2.9) \quad \begin{aligned} 0 &= \lim_{h \rightarrow 0} \int_S \int_{-h_0/2}^{h_0/2} \det(\text{Id} + th/h_0\Pi) \left[ (Q^h)^T R^h E^h \right]_{TS} : [\nabla_{tan}\phi(x)(\text{Id} + th/h_0\Pi)^{-1}] \, dt dx \\ &= \int_S \int_{-h_0/2}^{h_0/2} E_{TS} : \nabla_{tan}\psi(x) \, dt dx = \int_S \mathcal{L}_2\left(x, B_{tan} - \frac{\kappa}{2}(A^2)_{tan}\right) : [\nabla\phi(x)]_{tan} \, dx \\ &= \int_S \mathcal{L}_2\left(x, B_{tan} - \frac{\kappa}{2}(A^2)_{tan}\right) : \text{sym } \nabla\phi(x) \, dx \end{aligned}$$

where we have used Lemma 2.2 (i), Lemma 2.3 and (2.7). Therefore, by density of  $\{\text{sym } \nabla\phi\}$  in the space  $\mathcal{B}$ , (1.13) follows immediately.

**Proof of the second Euler-Lagrange equation (1.14).**

Let  $\tilde{V} \in \mathcal{V}$  and denote by  $\tilde{A}$  the skew-symmetric matrix field representing  $\nabla\tilde{V}$ , as in (1.5).

**1.** We now apply (2.4) to a variation of the form:  $\psi(x + t\vec{n}) = t\tilde{A}\vec{n}(x)$ . For simplicity, write  $\eta = \tilde{A}\vec{n} \in W^{1,2}(S, \mathbb{R}^3)$ . Upon dividing (2.4) by  $h$  and passing to the limit, we obtain:

$$(2.10) \quad \begin{aligned} 0 &= \lim_{h \rightarrow 0} \left[ \int_S \int_{-h_0/2}^{h_0/2} \det(\text{Id} + th/h_0\Pi) \left[ (Q^h)^T R^h tE^h \right]_{TS} : [\nabla_{tan}\eta(x)(\text{Id} + th/h_0\Pi)^{-1}] \, dt dx \right. \\ &\quad + \frac{h_0}{h} \int_S \int_{-h_0/2}^{h_0/2} ((Q^h)^T R^h E^h \vec{n}) \, \eta(x) \, dt dx \\ &\quad \left. + \int_S \int_{-h_0/2}^{h_0/2} (t \, \text{trace } \Pi + t^2 h/h_0 \det \Pi) ((Q^h)^T R^h E^h \vec{n}) \, \eta(x) \, dt dx \right], \end{aligned}$$

where we used the identity:

$$\det(\text{Id} + th/h_0\Pi) = 1 + th/h_0 \text{trace } \Pi + t^2 h^2/h_0^2 \det \Pi.$$

The first term in (2.10), in view of Lemma 2.2 (ii), Lemma 2.3 and (2.8), converges to:

$$\begin{aligned} \int_S \int_{-h_0/2}^{h_0/2} tE_{TS} : \nabla_{tan}\eta(x) \, dt dx &= \int_S \hat{E}_{tan} : (\nabla\eta(x))_{tan} \, dt dx \\ &= \frac{h_0}{12} \int_S \mathcal{L}_2(x, (\nabla(A\vec{n}) - A\Pi)_{tan}) : (\nabla\eta(x))_{tan} \, dx. \end{aligned}$$

In turn, the third term in (2.10) converges to 0. This is because  $(Q^h)^T R^h E^h \vec{n}$  converge weakly in  $L^2(S^{h_0}, \mathbb{R}^3)$  to  $E\vec{n} = 0$ , by Lemma 2.2 (ii) and Lemma 2.3 (i). Summarizing, (2.10) yields:

$$(2.11) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_S \int_{-h_0/2}^{h_0/2} ((Q^h)^T R^h E^h \vec{n}) \, \tilde{A}\vec{n} \, dt dx = -\frac{1}{12} \int_S \mathcal{L}_2(x, (\nabla(A\vec{n}) - A\Pi)_{tan}) : (\nabla(\tilde{A}\vec{n}))_{tan} \, dx.$$



**2.** Now, apply (2.4) to the variation  $\psi(x + t\vec{n}) = \tilde{V}(x)$ , and pass to the limit after dividing both sides of (2.4) by  $h^2$ :

$$\begin{aligned}
 \int_S f(x) \cdot \bar{Q} \tilde{V}(x) \, dx &= \lim_{h \rightarrow 0} \int_S f(x) \cdot Q^h \tilde{V}(x) \, dx \\
 &= \lim_{h \rightarrow 0} \left[ \int_S \int_{-h_0/2}^{h_0/2} \left[ \frac{1}{h} ((Q^h)^T R^h - \text{Id}) E^h \right]_{TS} : \left[ \tilde{A}(x)_{TS} (\text{Id} + th/h_0 \text{adj } \Pi) \right] \, dt dx \right. \\
 &\quad \left. + \int_S \int_{-h_0/2}^{h_0/2} \frac{1}{h} E_{TS}^h : \left[ \tilde{A}(x)_{TS} (\text{Id} + th/h_0 \text{adj } \Pi) \right] \, dt dx \right] \\
 &:= \lim_{h \rightarrow 0} [I_h + II_h],
 \end{aligned} \tag{2.12}$$

where we used the definition of the adjoint matrix:

$$\det(\text{Id} + th/h_0 \Pi) (\text{Id} + th/h_0 \Pi)^{-1} = \text{adj } (\text{Id} + th/h_0 \Pi) = \text{Id} + th/h_0 \text{adj } \Pi.$$

Notice that, by Lemma 2.1 (ii) and (1.4), the matrix field:

$$1/h((Q^h)^T R^h - \text{Id}) = (\sqrt{e^h}/h^2)h/\sqrt{e^h}((Q^h)^T R^h - \text{Id})$$

converges to  $\kappa A$ , weakly in  $W^{1,2}(S)$  and hence strongly in  $L^2(S)$ . Hence, by the weak convergence of  $E^h$  to  $E$  and the uniform convergence of  $(\text{Id} + th/h_0 \text{adj } \Pi)$  to  $\text{Id}$ , the first term of (2.12) converges to:

$$\begin{aligned}
 \lim_{h \rightarrow 0} I_h &= \kappa \int_S \int_{-h_0/2}^{h_0/2} (AE)_{TS} : \tilde{A}_{TS} = \kappa \int_S (A\bar{E})_{TS} : \tilde{A}_{TS} = \kappa \int_S (A\bar{E}) : \tilde{A} \\
 &= -\kappa \int_S \bar{E} : (A\tilde{A}) = -\kappa \int_S \bar{E}_{tan} : (A\tilde{A})_{tan} \\
 &= -\kappa \int_S \mathcal{L}_2 \left( x, B_{tan} - \frac{\kappa}{2} (A^2)_{tan} \right) : (A\tilde{A})_{tan} \, dx,
 \end{aligned} \tag{2.13}$$

where we also have used Lemma 2.3 and (2.7).

**3.** Towards finding the limit of  $II_h$  in (2.12), consider first the contribution of the tangential minors. By Lemma 2.4 (ii) and since  $\tilde{A} \in L^p(S^{h_0})$  for all  $p \geq 1$ , one observes that:

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_S \int_{-h_0/2}^{h_0/2} \text{skew } E_{tan}^h : \tilde{A}_{tan} = 0. \tag{2.14}$$

Hence:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \int_S \int_{-h_0/2}^{h_0/2} \frac{1}{h} E_{tan}^h : \left[ \tilde{A}(x)_{tan} (\text{Id} + th/h_0 \text{adj } \Pi) \right] \, dt dx \\
 &= \frac{1}{h_0} \lim_{h \rightarrow 0} \int_S \int_{-h_0/2}^{h_0/2} t E_{tan}^h : \left[ \tilde{A}_{tan} \text{adj } \Pi \right] = \frac{1}{h_0} \lim_{h \rightarrow 0} \int_S \hat{E}_{tan} : \left[ \tilde{A}_{tan} \text{adj } \Pi \right] \\
 &= -\frac{1}{h_0} \lim_{h \rightarrow 0} \int_S \hat{E}_{tan} : (\tilde{A}_{tan} \Pi)^T = -\frac{1}{12} \int_S \mathcal{L}_2 \left( x, (\nabla(A\vec{n}) - A\Pi)_{tan} \right) : (\tilde{A}\Pi)_{tan} \, dx,
 \end{aligned} \tag{2.15}$$

where we have used (2.8) and Lemma 2.3 (ii), combined with the following formula, which can be easily checked for  $\tilde{A}_{tan} \in so(2)$ :

$$\tilde{A}_{tan} \text{adj } \Pi = -(\tilde{A}_{tan} \Pi)^T.$$

Further, by (2.11):

$$\begin{aligned}
(2.16) \quad & \lim_{h \rightarrow 0} \int_S \int_{-h_0/2}^{h_0/2} \frac{1}{h} \left( (E^h)^T \vec{n} \right) \left( (\tilde{A})^T \vec{n} \right) dt dx \\
&= - \lim_{h \rightarrow 0} \frac{1}{h} \int_S \int_{-h_0/2}^{h_0/2} ((Q^h)^T R^h E^h \vec{n}) (\tilde{A} \vec{n}) \\
&\quad + \lim_{h \rightarrow 0} \int_S \int_{-h_0/2}^{h_0/2} \left[ \frac{1}{h} ((Q^h)^T R^h - \text{Id}) (E^h \vec{n}) \right] (\tilde{A} \vec{n}) \\
&\quad + 2 \lim_{h \rightarrow 0} \int_S \int_{-h_0/2}^{h_0/2} \left[ \frac{1}{h} (\text{skew } E^h) \vec{n} \right] (\tilde{A} \vec{n}) \\
&= \frac{1}{12} \int_S \mathcal{L}_2(x, (\nabla(A\vec{n}) - A\Pi)_{tan}) : (\nabla(\tilde{A}\vec{n}))_{tan} dx.
\end{aligned}$$

Indeed,  $1/h((Q^h)^T R^h - \text{Id})$  converges to  $\kappa A$  weakly in  $L^4(S)$  while  $\tilde{A}\vec{n} \in L^4(S)$  and  $\bar{E}^h \vec{n}$  converges to 0 weakly in  $L^2(S)$ . Therefore the second term in (2.16) converges to 0. The last limiting term there vanishes as well, by Lemma 2.4 (ii) as in (2.14).

Finally, we have:

$$(2.17) \quad \lim_{h \rightarrow 0} \frac{1}{h_0} \int_S \int_{-h_0/2}^{h_0/2} \left( \vec{n}^T t E^h \right)_{tan} (\text{adj } \Pi) \left( (\tilde{A})^T \vec{n} \right)_{tan} dt dx = 0,$$

because  $(\hat{E}^h)^T \vec{n}$  converges to 0 weakly in  $L^2(S)$  by Lemma 2.2 (i) and Lemma 2.3.

Adding now (2.15), (2.16) and (2.17) we obtain:

$$(2.18) \quad \lim_{h \rightarrow 0} II_h = \frac{1}{12} \int_S \mathcal{L}_2(x, (\nabla(A\vec{n}) - A\Pi)_{tan}) : (\nabla(\tilde{A}\vec{n}) - \tilde{A}\Pi)_{tan} dx.$$

Together with (2.12) and (2.13), the formula (2.18) implies (1.14). ■

### 3. CONVERGENCE OF CRITICAL POINTS OF THE 3D ENERGY FUNCTIONALS

In this section we prove Theorem 1.2. Proceeding as in the proof of Theorem 1.1, one needs to exchange the expression  $\int_{S^h} DW(\nabla u^h) : \nabla \phi^h$  by that of  $\lim_{\epsilon \rightarrow 0} \int_{S^h} \frac{1}{\epsilon} [W(\nabla u^h + \epsilon \nabla \phi^h) - W(\nabla u^h)] dz$ . As shown below, the error given by the difference of these two quantities, converges to 0 as  $h \rightarrow 0$ , after an appropriate scaling by powers of  $h$  and  $\sqrt{e^h}$  and along the variations  $\phi^h$  used in the proof of Theorem 1.1.

We first prove a more general lemma, in which we derive the optimal asymptotic properties a sequence  $u^h$  must satisfy in order that the conclusions of Theorem 1.1 hold true. These properties will later be established for the critical points (1.9) of the functional  $J^h$ .

**Lemma 3.1.** *Assume (1.2) and (1.8). Let  $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$  be a sequence of deformations, satisfying:  $E^h(u^h) \leq C e^h$  where the scaling  $e^h$  is as in (1.4). For every  $\psi \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$ , consider the rescaled variations  $\phi^h$  given by (2.1) and define the corresponding error terms:*

$$\mathcal{E}_h(\psi) = \int_{S^h} DW(\nabla u^h) : \nabla \phi^h - \int_{S^h} f^h \phi^h.$$

Then all assertions of Theorem 1.1 hold, provided that:

- (i)  $\lim_{h \rightarrow 0} \frac{1}{\sqrt{e^h}} \mathcal{E}_h(\psi) = 0$ , for all  $\psi \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$ ,

- (ii)  $\lim_{h \rightarrow 0} \frac{1}{h\sqrt{e^h}} \mathcal{E}_h(\psi) = 0$ , for all  $\psi$  of the form  $\psi(x + t\vec{n}) = \phi(x)$ ,  $\phi \in W^{1,2}(S, \mathbb{R}^3)$ , and all  $\psi$  of the form  $\psi(x + t\vec{n}) = t\tilde{A}\vec{n}(x)$  with  $\tilde{A}$  given as in (1.5) for some  $\tilde{V} \in \mathcal{V}$ ,
- (iii)  $\lim_{h \rightarrow 0} \frac{1}{h^2\sqrt{e^h}} \mathcal{E}_h(\psi) = 0$ , for all  $\psi$  of the form  $\psi(x + t\vec{n}) = \tilde{V}(x)$  with  $\tilde{V} \in \mathcal{V}$ .

*Proof.* The proof follows by a direct inspection of the proof of Theorem 1.1. Indeed, (i) is needed to derive (2.6), (ii) serves for getting (2.9) and (2.11), through (2.10), while (iii) implies (2.12). ■

Theorem 1.2 is now a consequence of the following observation:

**Lemma 3.2.** *Assume (1.2) and (1.8). Let  $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$  satisfy:  $E^h(u^h) \leq Ce^h$  with the scaling  $e^h$  is as in (1.4). If (1.9) holds then the conditions (i), (ii), (iii) in Lemma 3.1 are fulfilled.*

*Proof.* In view of (1.9), it is enough to prove that (i), (ii), and (iii) in Lemma 3.1 hold with  $\mathcal{E}_h(\psi)$  replaced by a more convenient error quantity:

$$\mathcal{E}'_h(\psi) = \lim_{\epsilon \rightarrow 0} \mathcal{E}'_{h,\epsilon}(\psi), \quad \mathcal{E}'_{h,\epsilon}(\psi) = \int_{S^h} \left[ \frac{1}{\epsilon} \left( W(\nabla u^h + \epsilon \nabla \phi^h) - W(\nabla u^h) \right) \right] - DW(\nabla u^h) : \nabla \phi^h \, dz,$$

and the rescaled variations  $\phi^h$  given by (2.1).

Define the good sets:  $\Omega_{h,\epsilon} = \{z \in S^h; \text{dist}(\nabla u^h(z), SO(3)) < \delta \text{ and } \epsilon |\nabla \phi^h(z)| < \delta\}$ , with  $\delta > 0$  small enough for  $W$  to be  $C^2$  in the open neighborhood  $\{F \in \mathbb{R}^{3 \times 3}; \text{dist}(F, SO(3)) < 3\delta\}$ . We will estimate  $\mathcal{E}'_{h,\epsilon}(\psi)$  by writing it as a sum of two integrals: one over  $\Omega_{h,\epsilon}$  and the other over  $S^h \setminus \Omega_{h,\epsilon}$ . Apply the mean value theorem to the continuous function  $DW$  in the first integral, while in the second integral we use the assumption (1.8):

$$\begin{aligned} |\mathcal{E}'_{h,\epsilon}(\psi)| &= \left| \int_{S^h} \int_0^1 \left[ DW(\nabla u^h + \epsilon s \nabla \phi^h) - DW(\nabla u^h) \right] \, ds : \nabla \phi^h \, dz \right| \\ &\leq C\epsilon \int_{\Omega_{h,\epsilon}} |\nabla \phi^h| \, dz \\ &\quad + C \int_{S^h \setminus \Omega_{h,\epsilon}} \int_0^1 \left[ \text{dist}(\nabla u^h + \epsilon s \nabla \phi^h, SO(3)) + \text{dist}(\nabla u^h, SO(3)) \right] \, ds \cdot |\nabla \phi^h| \, dz \\ &\leq C\epsilon \int_{\Omega_{h,\epsilon}} |\nabla \phi^h| + C \int_{S^h \setminus \Omega_{h,\epsilon}} \text{dist}(\nabla u^h, SO(3)) |\nabla \phi^h| + C\epsilon \int_{S^h \setminus \Omega_{h,\epsilon}} |\nabla \phi^h|^2. \end{aligned} \tag{3.1}$$

We see that the first and the third term above converge to 0 as  $\epsilon \rightarrow 0$ . To treat the second term, notice that by (1.2), the energy bound, and (2.3):

$$\begin{aligned} \int_{S^h \setminus \Omega_{h,\epsilon}} \text{dist}(\nabla u^h, SO(3)) |\nabla \phi^h| &\leq C \int_{S^h \setminus \Omega_{h,\epsilon}} W(\nabla u^h)^{1/2} |\nabla \phi^h| \\ &\leq C \left[ hE^h(u^h) \right]^{1/2} \|\nabla \phi^h\|_{L^2(S^h \setminus \Omega_{h,\epsilon})} \leq Ch^{1/2} \sqrt{e^h} \|\nabla \phi^h\|_{L^2(S^h \setminus \Omega_{h,\epsilon})} \end{aligned} \tag{3.2}$$

Observe also that:

$$\int_{S^h \setminus \Omega_{h,\epsilon}} |\nabla \phi^h|^2 \leq C \int_{S^{h_0} \setminus \omega_{h,\epsilon}} \left[ h |\nabla_{\tan} \psi|^2 + \frac{1}{h} |\partial_{\vec{n}} \psi|^2 \right] \, dz,$$

where the set  $\omega_{h,\epsilon} = S^{h_0} \setminus \{x + t\vec{n}; x + th/h_0\vec{n} \in \Omega_{h,\epsilon}\}$ . Its measure can be estimated as:

$$\begin{aligned}
 |\omega_{h,\epsilon}| &\leq C/h |S^h \setminus \Omega_{h,\epsilon}| \\
 (3.3) \quad &\leq C/h \left\{ |z \in S^h; \text{dist}(\nabla u^h(z), SO(3)) \geq \delta| + |z \in S^h; \epsilon |\nabla \phi^h(z)| \geq \delta| \right\} \\
 &\leq C/h \left\{ \int_{S^h} W(\nabla u^h) + \epsilon^2 \int_{S^h} |\nabla \phi^h|^2 \right\}.
 \end{aligned}$$

In particular:

$$(3.4) \quad \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} |\omega_{h,\epsilon}| = 0.$$

We now prove (i). Passing to the limit in (3.1) and (3.2), and using (3.3) with (3.4) we obtain:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{e^h}} |\mathcal{E}'_{h,\epsilon}(\psi)| &\leq C \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} h^{1/2} \|\nabla \phi^h\|_{L^2(S^h \setminus \Omega_{h,\epsilon})} \\
 &\leq \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left( h \|\nabla_{\tan} \psi\|_{L^2(S^{h_0})} + \|\partial_{\vec{n}} \psi\|_{L^2(\omega_{h,\epsilon})} \right) = 0.
 \end{aligned}$$

To prove (ii), consider:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{h\sqrt{e^h}} |\mathcal{E}'_{h,\epsilon}(\psi)| &\leq C \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} h^{-1/2} \|\nabla \phi^h\|_{L^2(S^h \setminus \Omega_{h,\epsilon})} \\
 &\leq C \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left( \|\nabla_{\tan} \psi\|_{L^2(\omega_{h,\epsilon})} + \frac{1}{h} \|\partial_{\vec{n}} \psi\|_{L^2(\omega_{h,\epsilon})} \right).
 \end{aligned}$$

The first limit above is 0 by (3.4). Concerning the second term, it may be dropped for  $\psi(x + t\vec{n}) = \phi(x)$ , while in the other case when  $\phi(x + t\vec{n}) = t\tilde{A}\vec{n}(x)$  we have  $\partial_{\vec{n}}\psi = \tilde{A}\vec{n} \in W^{1,2}(S)$  and hence:

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{h} \|\partial_{\vec{n}} \psi\|_{L^2(\omega_{h,\epsilon})} \leq \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} C/h |\omega_{h,\epsilon}|^{1/3} \|\partial_{\vec{n}} \psi\|_{L^6(S)} \leq \lim_{h \rightarrow 0} C/h (e^h)^{1/3} = 0,$$

in view of (1.4).

To prove (iii) for  $\psi(x + t\vec{n}) = \tilde{V}(x)$ , recall that  $\nabla \tilde{V} \in W^{1,2}(S, \mathbb{R}^3)$  and write:

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{h^2 \sqrt{e^h}} |\mathcal{E}'_{h,\epsilon}(\psi)| \leq \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} C/h \|\nabla \tilde{V}\|_{L^2(\omega_{h,\epsilon})} \leq \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} C/h |\omega_{h,\epsilon}|^{1/3} \|\nabla \tilde{V}\|_{L^6(S)} = 0,$$

as before. This achieves the proof. ■

#### 4. THE LIMITING ROTATIONS $\bar{Q}$

In this section we will derive the third Euler-Lagrange equation (after the first two (1.13) and (1.14)), corresponding to variation in  $\bar{Q} \in SO(3)$ , and under certain nondegeneracy condition. We first notice that the limiting  $\bar{Q}$  necessarily satisfies the constraint of the average torque:

$$(4.1) \quad \tau(\bar{Q}) = \int_S f \times \bar{Q}x \, dx = 0.$$

The main difficulty arises now from the fact that the variations must be taken inside  $SO(3)$  in a way that this constraint remains satisfied. Assuming that such variations exist, we establish the limit equation under the additional condition that  $Q^h$  approach  $\bar{Q}$  along a direction  $U \in T_{\bar{Q}}SO(3)$  for which  $\partial_U \tau(\bar{Q}) \neq 0$ .

In what follows, the crucial role is played by the function  $g(Q) = \int_S f \cdot Qx \, dx$  defined on  $SO(3)$ . Let  $B \in \mathbb{R}^{3 \times 3}$  be such that:  $g(Q) = B : Q$ , for all  $Q \in SO(3)$ .

**Lemma 4.1.** *Assume the hypothesis of Theorem 1.1 or Theorem 1.2. Then the limit  $\bar{Q} \in SO(3)$  of  $Q^h$  must satisfy:*

$$(4.2) \quad \int_S f \cdot \bar{Q} F x \, dx = 0 \quad \forall F \in so(3),$$

or equivalently (4.1). Another equivalent formulation of (4.2) is:  $\text{skew}(\bar{Q}^T B) = 0$ .

*Proof.* First, for any given  $H \in so(3)$ , consider the variation  $\phi^h = Hu^h$  in the equilibrium equation (1.10). Recalling that  $DW(\nabla u^h)(\nabla u^h)^T$  is symmetric (see the proof of Lemma 2.4) we obtain:

$$(4.3) \quad \int_{S^h} f^h \cdot Hu^h = \int_{S^h} DW(\nabla u^h) : H \nabla u^h = \int_{S^h} (DW(\nabla u^h)(\nabla u^h)^T) : H = 0.$$

Similarly, taking  $\phi^h = \frac{1}{\epsilon}(\exp(\epsilon H)u^h - u^h)$  in (1.9), by frame indifference of  $W$  we get:

$$\int_{S^h} f^h \cdot Hu^h = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{S^h} f^h \cdot (\exp(\epsilon H)u^h - u^h) = h \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J^h(\exp(\epsilon H)u^h) - J^h(u^h)) = 0.$$

Now, for any sequence of skew-symmetric matrices  $F^h$  we have:

$$(4.4) \quad \begin{aligned} \int_S f \cdot Q^h F^h V^h &= \frac{1}{he^h} \int_{S^h} f^h \cdot Q^h F^h ((Q^h)^T u^h - c^h - \text{id}) \\ &= \frac{1}{he^h} \int_{S^h} f^h \cdot (Q^h F^h (Q^h)^T) u^h - \frac{1}{he^h} \int_{S^h} f^h \, dz \cdot Q^h F^h c^h - \frac{1}{he^h} \int_{S^h} f^h \cdot Q^h F^h z \, dz \\ &= -\frac{h}{\sqrt{e^h}} \int_S f \cdot Q^h F^h x \, dx, \end{aligned}$$

where the first two terms in the second line above vanish by taking  $H = Q^h F^h (Q^h)^T \in so(3)$  in (4.3), and by the normalization of  $f^h$ . Passing to the limit with  $h \rightarrow 0$  in (4.4), where  $F^h = F$ , we see that:  $-\int_S f \cdot \bar{Q} F V = \lim_{h \rightarrow 0} h/\sqrt{e^h} \int_S f \cdot Q^h F x \, dx$ . This implies (4.2).

Clearly, (4.2) is also equivalent to  $0 = B : \bar{Q} F = \bar{Q}^T B : F$  for all  $F \in so(3)$ , which means exactly that  $\bar{Q}^T B$  is a symmetric matrix.

To prove the other equivalent formulation of (4.2), notice that:

$$\int_S f \cdot \bar{Q} F x = \int_S \bar{Q}^T f \cdot F x = -c_F \cdot \int_S \bar{Q}^T f \times x = -c_F \int_S f \times \bar{Q} x,$$

where  $c_F \in \mathbb{R}^3$  is such that  $Fx = c_F \times x$  for all  $x \in \mathbb{R}^3$ . Since there is a one to one correspondence between vectors  $c_F$  and skew matrices  $F$ , the proof is achieved.  $\blacksquare$

Define now the set of the rotation equilibria:

$$\mathcal{M} = \{\bar{Q} \in SO(3); \text{skew}(\bar{Q}^T B) = 0\}.$$

Our goal is to derive the third Euler-Lagrange equation, with respect to the variations of  $\bar{Q}$  in  $\mathcal{M}$ . For  $\bar{Q} \in \mathcal{M}$ , let  $F \in so(3)$  be such that:

$$\bar{Q} F = \lim_{n \rightarrow \infty} \frac{\bar{Q}_n - \bar{Q}}{\|\bar{Q}_n - \bar{Q}\|},$$

for some  $\bar{Q}_n \in \mathcal{M}$  converging to  $\bar{Q}$ . Clearly, the above implies that:

$$(4.5) \quad \text{skew}(F \bar{Q}^T B) = 0.$$

**Lemma 4.2.** *Under the hypothesis of Theorem 1.1 or Theorem 1.2, assume moreover that:*

$$\lim_{h \rightarrow 0} \frac{Q^h - \bar{Q}}{\|Q^h - \bar{Q}\|} = \bar{Q}H, \quad \text{with} \quad \text{skew}(H\bar{Q}^T B) \neq 0.$$

*Then for every  $F \in so(3)$  satisfying (4.5) there holds:*

$$\int_S f \cdot \bar{Q} F V \, dx = 0.$$

*Proof.* We will find a sequence  $F^h \in so(3)$ , converging to  $F$  and such that, for all  $h$ :

$$(4.6) \quad \int_S f \cdot Q^h F^h x \, dx = 0.$$

In view of (4.4) this will prove the lemma. Existence of such approximating sequence  $F^h$  is guaranteed by the assumed nondegeneracy condition:  $\text{skew}(H\bar{Q}^T B) \neq 0$ .

Firstly, notice that for  $Q^h \in \mathcal{M}$  one can take  $F^h = F$ . Otherwise, define:

$$F^h = F - \frac{(Q^h)^T B : F}{|\text{skew}((Q^h)^T B)|^2} \text{skew}((Q^h)^T B).$$

Then:

$$\begin{aligned} \int_S f \cdot Q^h F^h x \, dx &= B : Q^h F^h = (Q^h)^T B : F^h \\ &= (Q^h)^T B : F - \frac{(Q^h)^T B : F}{|\text{skew}((Q^h)^T B)|^2} (Q^h)^T B : \text{skew}((Q^h)^T B) = 0, \end{aligned}$$

and moreover:

$$\begin{aligned} \lim_{h \rightarrow 0} |F^h - F| &= \lim_{h \rightarrow 0} \frac{|(Q^h)^T B : F|}{|\text{skew}((Q^h)^T B)|} = \lim_{h \rightarrow 0} \frac{|(Q^h)^T B : F - \bar{Q}^T B : F|}{|\text{skew}((Q^h)^T B - \bar{Q}^T B)|} \\ &= \lim_{h \rightarrow 0} \left| \left( \frac{Q^h - \bar{Q}}{\|Q^h - \bar{Q}\|} \right)^T B : F \right| / \left| \text{skew} \left( \frac{Q^h - \bar{Q}}{\|Q^h - \bar{Q}\|} \right)^T B \right| = \frac{|H^T \bar{Q}^T B : F|}{|\text{skew}(H^T \bar{Q}^T B)|} = 0. \end{aligned}$$

The last expression above equals to 0 because of the nullity of its numerator:

$$H^T \bar{Q}^T B : F = \bar{Q}^T B : HF = \bar{Q}^T B : (HF)^T = \bar{Q}^T B : FH = -F \bar{Q}^T B : H = 0,$$

where we have used that  $\bar{Q}^T B$  is symmetric and (4.5). ■

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